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SOME REMARKS ON THE DIFFERENTIAL 1-FORM

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Abstract. In this paper we will give some alternate proofs of some propositions about differential 1-form and path integration. We have noticed if for the given differential 1-form for a smooth function, in some subsets of the plane, then there does not exist a smooth function such that its differential is equal to the 1-form. In the end of this paper we construct subdivision and proof that for every closed 1-form, the path integral is equal to the sum of the endpoints in subintervals.

1. Introduction and auxiliary facts

We denote with U an open set in the plane $\mathbb{R}^2$.

Definition 1.1. A smooth function or C^∞ function on U is a function $f : U \to R$ such that all partial derivatives of all order exists and are continuous.

A differential 1-form, or just a 1-form, on U is given by a pair of smooth functions p and q on U. We will denote a 1-form by w and we will write $w = pdx + qdy$.
By a smooth path or just path in U, we mean a mapping $\gamma : [a, b] \to U$ from a bounded interval into U that is continuous on $[a, b]$ and differentiable in the open interval (a, b). So $\gamma(t) = (x(t), y(t))$ and $\gamma(a), \gamma(b)$ are called the endpoints.

With $w = pdx+qdy$ as above and γ a path given by the pair of functions $\gamma(t)$, the integral $\int_w = \int_a^b \left(p(x(t), y(t))\frac{dx}{dt} + q(x(t), y(t))\frac{dx}{dt}\right) dt$.

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We write $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ for this 1-form and say that w is the differential of f if $w = df$.

Proposition 1.2. *$df = dg$ on U if and only if $f - g$ is locally constant on U*

Proposition 1.3. *If γ is a segmented path in U from P to Q and $w = df$ in U then*

$$\int_\gamma w = f(Q) - f(P).$$

Proposition 1.4. *Let U be a product of two open finite or infinite intervals, i.e., $U = \{(x,y) : a < x < b \text{ and } c < y < d\}$, with $-\infty \leq a < b \leq \infty$ and $-\infty \leq c < d \leq \infty$. If w is any 1-form on U such that $dw = 0$, then there is function f on U with $w = df$.*

*A 1-form w is called **closed** if $dw = 0$ and is called **exact** if $w = df$.*

2. Main results

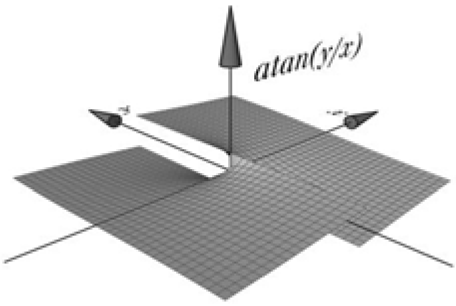
Proposition 2.1. *There is no smooth function $g \in C^\infty$ such that $dg = w_\theta$ on U where $w_\theta = \frac{-ydx+xdy}{x^2+y^2}$ and U is*

a) the upper half plane
b) the union of the upper half plane and the right half plane
c) the complement of the negative x-axis.

Proof. If we take differential to the function $f(x,y) = \arctan(\frac{y}{x})$ we have $df = \frac{-ydx+xdy}{x^2+y^2}$. Let we denote this differential with w_θ. The function $f(x,y) = \arctan(\frac{y}{x})$ is not differential in $x = 0$ (because is composition of $z = \frac{y}{x}$ defined everywhere expect in the point $x = 0$, the function arctan is continuous). If there exist $g \in C^\infty$ such that $dg = w_\theta$ on U then the integral $\int_\gamma w_\theta$ depends on endpoints for every path γ on U (from Proposition 1.3), so we have

$$\int_\gamma w_\theta = f\left(\gamma(b) - \gamma(a)\right),$$

where $\gamma : [a,b] \to U, \gamma(t) = (x(t), y(t))$. The graph of the function $f(x,y) = \arctan\frac{y}{x}$ is:



arctan(y/x) graph.

From the graph we can decide that U can be the right and the left half plane, then there exist $g \in C^\infty$ such that $dg = w_\theta$ on U, while $f(x, y) = \arctan \frac{y}{x}$ is not defined in $x = 0$, while the surface is divided by $x = 0$ which means according to the y-axis. If U is the set in the cases a), b) or c) we can not find such a function. If we suppose that there exists a function $g \in C^\infty$, such that $dg = w\theta$ on U we obtain the case

a) Let $\gamma(t) = (\cos(t), \sin(t)), 0 \leq t \leq \pi$ is path on U, then
$$\int_\gamma w_\theta = \int_0^\pi df = \int_0^\pi \frac{-ydx + xdy}{x^2 + y^2} = \int_0^\pi 1dt = \pi$$
On the other hand we have $\gamma(0) = (1, 0), \gamma(\pi) = (-1, 0)$. So we obtain
$$f(\gamma(\pi)) - f(\gamma(0)) = f(-1, 0) - f(1, 0) = \arctan 0^\circ - \arctan 0^\circ = 0$$
While $\int_\gamma w_\theta = \pi \neq 0 = f(\gamma(\pi)) - f(\gamma(0))$. So we get that there is no function $g \in C^\infty$, such that $dg = w_\theta$ on U.

b) Let $\gamma(t) = (\cos(t), \sin(t)), -\frac{\pi}{2} \leq t \leq \pi$ is path on U, then
$$\int_\gamma w_\theta = \int_{-\frac{\pi}{2}}^\pi df = \int_{-\frac{\pi}{2}}^\pi \frac{-ydx + xdy}{x^2 + y^2} = \int_{-\frac{\pi}{2}}^\pi 1dt = \frac{3\pi}{2}$$
On the other hand we have $\gamma(-\frac{\pi}{2}) = (0, -1), \gamma(\pi) = (-1, 0)$. So we obtain
$$f(\gamma(\pi)) - f\left(\gamma(-\frac{\pi}{2})\right) = f(-1, 0) - f(0, -1) =$$
$$= \arctan \frac{0}{-1} - \arctan \frac{-1}{0} = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}.$$
So there is no function $g \in C^\infty$, such that $dg = w_\theta$ on U, where U is the union of the upper and the right half plane.

c) Similarly there is no function $g \in C^{\infty}$. Here easy we can take $\gamma(t) = (\cos(t), \sin(t)), -\frac{3\pi}{4} \leq t \leq \frac{3\pi}{4}$ on the other hand from the graph we have that the surface is divided from the y-axis).

□

Proposition 2.2. *There exists a function on* $\mathbb{R}^2/\{(0,0)\}$*, such that* $w = \frac{xdx+ydy}{(x^2+y^2)^2}$ *represents its differential.*

Proof. If the differential form $w = \frac{xdx+ydy}{(x^2+y^2)^2}$ can be written as the differential of function f on $\mathbb{R}^2/\{(0,0)\}$, than we have $df = w$.
If we integrate $df = w$, we obtain

$$f = \int \frac{xdx + ydy}{(x^2+y^2)^2} = \frac{1}{2}\int \frac{d(x^2+y^2)}{(x^2+y^2)^2} = -\frac{1}{2}\frac{1}{x^2+y^2}$$

So $f(x,y) = -\frac{1}{2(x^2+y^2)}$ is smooth function on $\mathbb{R}^2/\{(0,0)\}$ that fulfills the required condition. □

Proposition 2.3. *Let w be a closed 1-form on U and $\gamma : [a,b] \to U$, is a smooth path, then there exists a subdivision $a = t_0 < t_1 < ... < t_n = b$ and a collection of $U_i, i = 1,2,...,n$ where U_i is an open subset of U so that γ maps $[t_{i-1}, t_i]$ into U_i, and the restriction of w to U_i is the differential of a function f_i. For $P_i = \gamma(t_i)$ and for any such choices, the following relation is valid*

$$\int_{\gamma} w = \sum_{i=1}^{n} [f_i(P_i) - f_i(P_{i-1})]$$

Proof. For any point $P \in \gamma([a,b])$, we choose its neighborhood U_P on which the restriction of w is exact. It is clear that $\gamma^{-1}(U_P)$ form an open covering of the compact interval $[a,b]$, so the finite number of them cover the interval. From this it is not hard to construct the subdivision.
From $[a,b]$ compact set and $\gamma^{-1}(U_P)$ open covering then exist $\varepsilon > 0$ such that for every compact subset of $K = [a,b]$ with diameter less then ε , and is subset of any open set of covering.
Let we denote with A_n the segments of subdivision of $[a,b]$.
If it is not true, let us suppose that for A_n subset of $[a,b]$ with diameter less then $\frac{1}{n}$, $\left(d(A_n) < \frac{1}{n}\right)$ and is not a subset of any open set of covering $\gamma^{-1}(U_P)$.
From $[a,b]$ compact set, every infinitely subset have at least one boundary value, for example A. Let Ω is open set from open covering, let $A \in \Omega, r > 0$.

Then each point of $B\left(A, \frac{r}{2}\right)$ is element of Ω. We obtain $B\left(A, \frac{r}{2}\right) \cup A_n \neq \emptyset$ for infinitely terms, which is a contradiction.
So we have that every subset of $[a,b]$ with diameter $\varepsilon > 0(\varepsilon = \frac{1}{n})$, is subset of any open set of covering.
If we fix such a subdivision, choose one of these open sets U_P which contain $\gamma(A_i)$ and denote them with U_i and f_i in U_i such that $df_i = w$ in U_i.
Let $\gamma_i : [t_{i-1}, t_i] \to U_i$ be the restriction of γ in A_i. From $df_i = w$ in U_i (from Proposition 1.3 we have that integral depends from the end points).
So we have that

$$\begin{aligned}\int_\gamma w = \int_{\gamma_1} w + \int_{\gamma_2} w + ... + \int_{\gamma_n} w =\\ = \sum_{i=1}^{n} [f_i(\gamma_i(t_i)) - f_i(\gamma_i(t_{i-1}))] =\\ = \sum_{i=1}^{n} [f_i(P_i) - f_i(P_{i-1})].\end{aligned}$$

□

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